# Analysis II

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**Abstract**

[1]These notes will prove, among other things, the fundamental theorem of calculus, the theorems of Arzela-Ascoli and Dini, the existence of an everywhere-continuous nowhere-differentiable function, the Stone-Weierstrass theorem for general algebras of functions, the summation of the Bessel series $\sum 1/n^2 = \pi^2/6$ using Fourier analysis, and the equivalence of the inverse and implicit function theorems in $\mathbb{R}^n$.

[1]The content of these notes is largely based on *Principles of Mathematical Analysis* byWalter Rudin.

# Contents

# Chapter 1

# The Riemann-Stieltjes Integral

## 1.1 Definition and Existence of the Integral

Before embarking on a serious treatment of the theorems of integral calculus, it is prudent to construct a flexible integral defined on interval subsets of the real line that can handle the pathologies not present in "well-behaved" functions. This integral should be defined for any bounded function with countably many discontinuities. In particular, the presence of a removable discontinuity should not change the value of the integral, that is to say, the integral of a discontinuous function should be equal to that of its continuous extension. These requirements are lax enough to allow us to integrate a whole host of misbehaved functions, such as the infinitely oscillatory $x\sin(x)$ over the interval $[0,1]$.

**Definition.** A partition $P$ of the interval $[a,b]$ is a set of finitely many points $\{x_n\}$ with

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

Assume $f$ is a bounded function on the interval $[a,b]$, i.e. $m \leq f(x) \leq M$ on $[a,b]$ for some numbers $m$ and $M$. We will define $L(f,P)$, the lower (Riemann) sum associated with $f$ and $P$, to be

$$L(f,P) = \sum_{i=0}^{n-1}(x_{i+1} - x_i) \inf_{x_i \leq x \leq x_{i+1}} f(x),$$

where

$$\inf_{x_i \leq x \leq x_{i+1}} f(x) = \inf\{f(x) : x_i \leq x \leq x_{i+1}\}.$$

This infinum is guaranteed to exist by the inequality $m \leq f$. Similarly, the upper sum of $f$ on the interval $[a, b]$ is defined to be

$$U(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup_{[x_i, x_{i+1}]} f.$$

It is easy to see that $L(f, P) \leq U(f, P)$ for any given partition $P$ of $[a, b]$.

**Definition.** We say that a partition $P^*$ refines (or is a refinement of) $P$ if $P^*$ contains all the points in $P$.

**Lemma.** *If $P^*$ refines $P$, then $L(P, f) \leq L(P^*, f)$ and $U(P^*, f) \leq U(P, f)$.*

*Proof.* It suffices to consider one interval within the partition $P$ and see what happens if we add an extra point. If $P$ is given by $a = x_0 < x_1 < x_2 < ...x_n = b$, then let $I = [x_0, x_1]$ denote the first interval of the partition. Suppose that the refinement $P^*$ contains a point $x^*$ situated between $x_0$ and $x_1$. The interval $I$ in $P^*$ is then broken into two intervals: $I_1 = [x_0, x^*]$ and $I_2 = [x^*, x_1]$. Then

$$\begin{aligned} L(f, P) &= (\text{length of } I) \times \inf_I f + ... \\ &\leq (\text{length of } I_1) \times \inf_{I_1} f + (\text{length of } I_2) \times \inf_{I_2} f + ... = L(f, P^*). \end{aligned}$$

The case for the upper sums is analogous. □

**Corollary.** *For every two partitions $P_1$ and $P_2$, we have $L(f, P_1) \leq U(f, P_2)$.*

*Proof.* Let $P^*$ be the partition that contains all points of $P_1$ and $P_2$. Then $P^*$ is a refinement of both $P_1$ and $P_2$. Now

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_2).$$

□

**Definition.** Assume $f$ is bounded on the interval $[a, b]$. Let the lower integral of $f$ be

$$\int_{\underline{a}}^{b} f = \sup\{L(f, P) : P \text{ is any partition of } [a, b]\}.$$

Similarly, the upper integral of $f$ is

$$\int_{a}^{\overline{b}} f = \inf\{U(f, P) : P \text{ is any partition of } [a, b]\}.$$

These integrals always exist (recall $m \leq f \leq M$) and $\int_{\underline{a}}^{b} f \leq \int_{a}^{\overline{b}} f$.

**Definition.** A function $f$ is called Riemann integrable ($f \in \mathfrak{R}$) on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$, with the integral being the common value.

When asking whether a given function is Riemann integrable, it is almost never practical to appeal to this definition as the construction of both the upper and lower integrals can be quite laborious. We will instead opt for a much more powerful test for integrability.

**Theorem 1** (Criterion for Integrability)**.** *A function $f$ is Riemann integrable if and only if for every $\epsilon > 0$, there exists a partition $P_\epsilon$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.*

*Proof.* "$\Leftarrow$": To prove $\underline{\int} = \overline{\int}$, it suffices to prove that $\overline{\int} - \underline{\int} < \epsilon$ for every $\epsilon > 0$. We are done by noting that

$$L(f, P_\epsilon) \leq \underline{\int} \leq \overline{\int} \leq U(f, P_\epsilon).$$

"$\Rightarrow$": Let the partitions $P_{1\epsilon}$ and $P_{2\epsilon}$ be such that $\int - L(f, P_{1\epsilon}) < \epsilon/2$ and $U(f, P_{2\epsilon}) - \int < \epsilon/2$. Consider $P_\epsilon$ the common refinement of $P_{1\epsilon}$ and $P_{2\epsilon}$. Then

$$\begin{aligned} \int - \epsilon/2 \leq L(f, P_{1\epsilon}) \leq L(f, P_\epsilon) \leq \\ \leq U(f, P_\epsilon) \leq U(f, P_{2\epsilon}) \leq \int + \epsilon/2. \end{aligned}$$

From this, it follows that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. ☐

We are ready to prove some basic facts about integrable functions, making use of our test above. The theorem below, for instance, will ensure the existence of $\int_0^1 x \sin(x) \mathrm{d}x$.

**Theorem 2.** *If $f$ is continuous on $[a,b]$, then $f \in \mathfrak{R}$.*

*Proof.* Fix $\epsilon > 0$. We need to find a partition $P_\epsilon$ with $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. We will appeal to the definition of continuity: $f$ is continuous at $x$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x)$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Further, $f$ is uniformly continuous (UC) on $[a,b]$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ for all $x, y \in [a,b]$. All continuous functions are uniformly continuous on an interval, so we can pick $\delta$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}$. Choose any partition $P_\epsilon$ containing exclusively intervals that are less than $\delta$. Then

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum \Delta x_i (\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f) < (\sum \Delta x_i) \frac{\epsilon}{b-a} = \epsilon.$$

☐

**Theorem 3.** *If $f$ is monotonic on $[a,b]$, then $f \in \mathfrak{R}$.*

*Proof.* To start, notice that if we choose $P$ such that $\Delta x_i = \frac{b-a}{n}$ is uniform for each $i$, then

$$\sum \Delta x_i (\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f) = \sum \Delta x_i (f(x_{i+1}) - f(x_i)) = \frac{b-a}{n}(f(b) - f(a)).$$

Without loss of generality, assume $f$ is monotonically increasing. We need to prove that for every $\epsilon > 0$, there exists a $P_\epsilon$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Given our scratch work above, it is clear that we would like $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$. Choose $n$ such that $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$, and pick $P_\epsilon$ to be the uniform partition with $n$ (equal sized) intervals. Then note that

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) = \sum \Delta x_i (f(x_{i+1}) - f(x_i)) &= \frac{b-a}{n} \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i)) \\ &= \frac{b-a}{n}(f(b) - f(a)) < \epsilon. \end{aligned}$$

$\square$

It is natural to ask ourselves how many discontinuities a function can have in its domain and still be integrable. We saw above that there are no restrictions in this regard if the function in question is monotonic. In general, however, the integral thus-defined will not be able to treat functions that are discontinuous "almost everywhere" (this language will become more meaningful later). For example, the everywhere-discontinuous Dirichlet function $f : [0,1] \to \mathbb{R}$ defined by

$$f = \begin{cases} 0, \text{if } x \in \mathbb{Q}, \\ 1, \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable. Functions that have a finite number of discontinuities, however, can still be treated.

**Theorem 4.** *Assume $f : [a,b] \to \mathbb{R}$ is bounded ($|f(x)| \le M$) and the set of discontinuities* $\mathrm{Disc}(f)$ *is finite. Then $f \in \mathfrak{R}$.*

*Proof.* Let $\epsilon$. We need to find the $P_\epsilon$.
Step I: For each $x \in \mathrm{Disc}(f)$, we cover $x$ with an open interval $I_x$ of length $\le \epsilon \,/\, 4M|\mathrm{Disc}(f)|$.
Step II: The set $[a,b] \setminus \cup_{x \in \mathrm{Disc}(f)} I_x$ is a compact set because the $I_x$ are open. The restriction of $f$ on this set is uniformly continuous. There is $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$

whenever $|x-y|<\delta$. Our $P_\epsilon$ will consist of the endpoints of the intervals $I_x$, together with $\delta$-separated points. Finally,

$$\begin{aligned}
U(P_\epsilon, f) - L(P_\epsilon, f) &= \sum_i (M_i - m_i)\Delta x_i \\
&= \sum_{i\colon [x_i, x_{i+1}] = \text{ some } I} + \sum_{\text{the other i's}} \\
&\leq 2M\Delta x_i + \sum_{\text{all } i} \frac{\epsilon}{2(b-a)}\Delta x_i \\
&\leq 2M\frac{\epsilon}{4M|\mathrm{Disc}(f)|} \cdot |\mathrm{Disc}(f)| + \frac{\epsilon}{2(b-a)} \cdot (b-a) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

### Ultimate Criterion for Riemann Integrability

A bounded $f$ is Riemann integrable if and only if $\mathrm{Disc}(f)$ has measure zero.

**Definition.** A set $S \subseteq \mathbb{R}$ has measure zero if for every $\epsilon > 0$, there exists a family of intervals $I$ that cover $S$ and $\sum_{\text{all } I} |I| < \epsilon$.

**Example.** If $S = \{x_n\}_{n\geq 1}$ is a countable set, then choose $I_n = (x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}})$. These intervals clearly cover $S$, and $\sum_{n\geq 1} \frac{\epsilon}{2^n} = \epsilon$. Thus, every countable set has measure zero.

**Example.** The Cantor set is uncountable, but still has measure zero. To see this, observe that

$$\begin{aligned}
|[0,1] \setminus \mathcal{C}| &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + ... = \sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} \\
&= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{3} \cdot \frac{1}{1-2/3} = 1.
\end{aligned}$$

**Theorem 5.** *Suppose $f \in \mathfrak{R}([a,b])$, $m \leq f(x) \leq M$, and $x \in [a,b]$. Assume $\phi : [m,M] \to \mathbb{R}$ is continuous. Then $h(x) = \phi(f(x)) \in \mathfrak{R}([a,b])$.*

*Proof.* □

# Exercises

1. Suppose $f \geq 0$, $f$ is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Hint: Try a proof by contradiction: assume $f(x_0) > 0$ for some $x_0$. What can you say about the upper (or lower) sums associated with any partition?

2. If $f(x) = 0$ for all irrational $x$, $f(x) = 1$ for all rational $x$, prove that $f \notin \mathfrak{R}$ on $[a, b]$ for any $a < b$.

3. (a) Construct a function $f : [0, 1] \to \mathbb{R}$ such that $f$ is not Riemann integrable on $[0, 1]$, but $f^2$ (i.e. $f$ squared) is Riemann integrable on $[0, 1]$. Justify your choice.

   (b) Assume $f : [0, 1] \to \mathbb{R}$ is such that $f^3$ is Riemann integrable. Prove that $f$ is also Riemann integrable.
   Hint for (b): Apply Theorem 6.11. What goes wrong when applying the same theorem to part (a)?

4. Suppose $f \in \mathfrak{R}$ on $[a, b]$ for every $b > a$ where $a$ is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. Such integrals are called *improper integrals*. If it also converges after $f$ has been replaced by $|f|$, it is said to converge *absolutely*.
Assume that $f(x) \geq 0$ and that $f$ decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series).

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous. For each $n \geq 1$, let $P_n$ be the partition of $[0, 1]$ into $n$ intervals of equal length $\frac{1}{n}$.

   (a) Prove that the sequence $a_n = U(P_n, f) - L(P_n, f)$ converges to zero.

   (b) Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) = \int_0^1 f(x)dx.$$

(c) Use (b) to find

$$\lim_{n\to\infty} \frac{1}{n^5} \sum_{i=1}^{n} i^4.$$

6. Assume $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Prove that for each $\epsilon > 0$ there is some $\delta > 0$ such that **for each** partition $P$ of $[a, b]$

$$P : a = x_0 < x_1 < ... < x_{i-1} < x_1 < ... < b = x_n$$

satisfying

$$x_i - x_{i-1} \leq \delta \text{ for each } 1 \leq i \leq n,$$

the following inequality holds

$$U(P, f) - L(P, f) < \epsilon.$$

Hint: Assume $|f(x)| \leq M$ on $[a, b]$. Fix $\epsilon$. Start with a partition $a = y_0 < y_1 < ... < y_m = b$ called $P_0$ such that

$$U(P_0, f) - L(P_0, f) < \epsilon/2.$$

Check the desired inequality holds for each partition $P : a = x_0 < ... < x_n = b$ with $\delta$ something like $\frac{\epsilon}{4Mm}$. Consider two types of intervals in $P$, somewhat similar to what we did in the proofs of theorems 6.10, 6.11. Note that each $y_j$ lies in some interval $[x_{i-1}, x_i]$.

7. Let $f, g : [0, 1] \to \mathbb{R}$ be defined as follows

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For a rational number in $[0, 1]$ we always pick the representation with $p, q$ positive and $p$ and $q$ relatively prime (so no common divisors).

For example, $0.5 = \frac{-1}{-2} = \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = ...$ and $g(0.5) = \frac{1}{2}$. Also, we may choose $g(0) = 0$, though this is irrelevant.

(a) Prove that $f$ and $g$ are Riemann integrable.

Hint: In Analysis I, we showed that $g$ is continuous on the rationals. Recall the criterium: a bounded function if Riemann integrable if and only if its discontinuities have measure zero.

(b) Prove that the composition function $h(x) = f(g(x))$ is not Riemann integrable on $[0, 1]$.

Comment: This shows the necessity of $f$ being continuous in Theorem 6.11 from Rudin.

## 1.2 Properties of the Integral

**Corollary.** *If $f, g \in \mathfrak{R}$, then*

1. *$f + g \in \mathfrak{R}$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

2. *$fg \in \mathfrak{R}$.*

*Proof.* 1.

$\square$

**Corollary** (Triangle Inequality)**.** 1. *If $f_1(x) \leq f_2(x)$, then*

$$\int_a^b f_1 \leq \int_a^b f_2.$$

2. *$f \in \mathfrak{R}$ implies that $|f| \in \mathfrak{R}$, and*

$$|\int_a^b f| \leq \int_a^b |f|.$$

*Proof.* 1.

$$\int_a^b f_1 \overset{\text{def}}{=} \sup\{L(f_1, P)\}$$
$$\int_a^b f_2 \overset{\text{def}}{=} \sup\{L(f_2, P)\}$$

Notice that $L(f_1, P) \leq L(f_2, P)$.

2. Since $\phi(x) = |x|$ is continuous, $|f| = \phi \circ f$ is Riemann integrable.

$\square$

## 1.3 Integration and Differentiation

This section provides the link between differential and integral calculus.

**Theorem 6.** *Assume $f \in \mathfrak{R}([a,b])$. Let $F(x) = \int_a^x f$ for some $x \in [a,b]$. Then*

*(a) $F$ is uniformly continuous on $[a,b]$.*

*(b) If $f$ is continuous at $x_0$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.*

*Proof.* (a) Let $x < y$. Then $F(y) - F(x) = \int_x^y f$. Assume $|f(t)| \le M$ for all $t \in [a,b]$. By the triangle inequality,

$$|F(y) - F(x)| = |\int_x^y f| \le \int_x^y |f| \le M(y-x).$$

Let $\epsilon > 0$. Pick $\delta = \epsilon/M$. If $|y-x| < \delta$, then $|F(x) - F(y)| < \epsilon$.

(b) Let $x_0 < t$. Then

$$\begin{aligned}
\left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| &= \left| \frac{\int_{x_0}^t f}{t - x_0} - f(x_0) \right| \\
&= \left| \frac{\int_{x_0}^t f(s)ds - f(x_0)(t - x_0)}{t - x_0} \right| \\
&= \frac{\left| \int_{x_0}^t f(s)ds - \int_{x_0}^t f(x_0)ds \right|}{|t - x_0|} \\
&= \frac{\left| \int_{x_0}^t (f(s) - f(x_0))ds \right|}{|t - x_0|} \\
&\le \frac{\int_{x_0}^t |(f(s) - f(x_0))| \, ds}{t - x_0} \\
&\le \max_{x_0 \le s \le t} |f(s) - f(x_0)|.
\end{aligned}$$

Since $f$ is continuous at $x_0$, $\lim_{t \to x_0} (\max_{x_0 \le s \le t} |f(s) - f(x_0)|) = 0$. Therefore,

$$\lim_{t \to x_0} \left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| = 0.$$

□

**Theorem 7** (The Fundamental Theorem of Calculus)**.** *Let $f \in \mathfrak{R}([a,b])$. Assume $f$ has an antiderivative $F$, i.e. $F$ is differentiable and $F'(x) = f(x)$. Then*

$$\int_a^b f = F(b) - F(a).$$

*Proof.* It suffices to prove that given any $\epsilon > 0$,

$$\left| \int_a^b f - (F(b) - F(a)) \right| < \epsilon.$$

Fix $\epsilon$. Let $P : a = x_0 < ... < x_n = b$ be such that $U(f,P) - L(f,P) < \epsilon$. Then

$$\begin{aligned} F(b) - F(a) = F(b) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + ... \\ ... + F(x_2) - F(x_1) + F(x_1) - F(a). \end{aligned}$$

Consider the interval $[x_{n-1}, b]$. The Mean Value Theorem furnishes a point $y_n \in [x_{n-1}, b]$ such that

$$\frac{F(b) - F(x_{n-1})}{b - x_{n-1}} = F'(y_n) = f(y_n).$$

Determining analogous $y_i$ for all intervals $[x_i, x_{i+1}]$, we find that

$$\begin{aligned} & F(b) - F(x_{n-1}) = \Delta x_n f(y_n) \\ & F(x_{n-1}) - F(x_{n-2}) = \Delta x_{n-1} f(y_{n-1}) \\ & ... \\ & F(x_1) - F(a) = \Delta x_1 f(y_1), \end{aligned}$$

so that

$$F(b) - F(a) = \Delta x_1 f(y_1) + ... + \Delta x_n f(y_n).$$

Thus $F(b) - F(a)$ is a Riemann sum with

$$L(f,P) \leq F(b) - F(a) \leq U(f,P).$$

From this, it follows that

$$\left| \int_a^b f - (F(b) - F(a)) \right| < \epsilon.$$

$\square$

**Theorem 8** (Integration by Parts)**.** *Let $F, G : [a,b] \to \mathbb{R}$ be differentiable functions. Assume $F' = f \in \mathfrak{R}$, $G' = g \in \mathfrak{R}$. Then*

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fg.$$

*Proof.* Apply FTC to $h = (FG)' = fG + Fg$:

$$\int_a^b h = (FG)(b) - (FG)(a).$$

$\square$

# Chapter 2

# Sequences and Series of Functions

## 2.1 Discussion of the Main Problem

## 2.2 Uniform Convergence

### Exercises

1. Let $f_n : E \to \mathbb{R}$ be a sequence of uniformly bounded functions. That means that there is some $M > 0$ such that $|f_n(x)| \leq M$ for each $n \geq 1$ and each $x \in E$. Assume also that $g_n : E \to \mathbb{R}$ is uniformly bounded.
   Assume that $f_n$ converges uniformly on $E$ and also that $g_n$ converges uniformly on $E$. Prove that the product $f_n g_n$ converges uniformly on $E$.

2. Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$.

   (a) Prove that the series converges for $x > 0$.

   (b) Prove that it does not converge uniformly on $(0, \infty)$.

   (c) Is $f$ bounded on $(0, \infty)$? Show rigorous details for your answer.

3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set $E$, but such that $\{f_n g_n\}$ does not converge uniformly on $E$ (of course, $\{f_n g_n\}$ must converge on $E$). Hint: Take $E = \mathbb{R}$, $f_n(x) = g_n(x) = x + \frac{1}{n}$.

4. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of $x$.

## 2.3 Uniform Convergence and Continuity

## Exercises

1. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function $f$ on a set $E$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.
Comment: $E$ here is any subset of any metric space $(X, d)$.

## 2.4 Uniform Convergence and Integration

## Exercises

1. Assume $f : [0,1] \to \mathbb{R}$ is Riemann integrable and continuous at 0.

   (a) Prove that the sequence $f_n(x) = f(x^n)$ converges pointwise on $[0,1]$.

   (b) Prove that the sequence $f_n$ converges uniformly if and only if $f(x) = f(0)$ for each $x \in [0,1)$.

   (c) Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)$.

## 2.5 Uniform Convergence and Differentiation

## Exercises

1. Prove that

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{5^n} \sin(4^n x)$$

is differentiable for each $x \in \mathbb{R}$ and its derivative equals

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \cos(4^n x).$$

Comment: Compare this with Thm 7.18.

Hint: Make the following argument rigorous. Need to prove that for each $\epsilon$ there is $\delta$ such that if $|t| < \delta$ then

$$\left|\frac{f(x+t) - f(x)}{t} - g(x)\right| < \epsilon.$$

This follows via the triangle inequality if we prove that

$$\sum_{n=0}^{\infty} \frac{1}{5^n} \left|\frac{\sin(4^n(x+t)) - \sin(4^n x)}{t} - 4^n \cos(4^n x)\right|.$$

Using the Mean Value Theorem, there is $y_{n,t}$ between $x$ and $x+t$ such that

$$\frac{\sin(4^n(x+t)) - \sin(4^n x)}{t} = 4^n \cos(4^n y_{n,t}).$$

To make the sum smaller than $\epsilon$, split it in two sums, and make each of the sums smaller than $\epsilon/2$. The first sum if finite, and will be small because of the continuity of $\cos(4^n x)$. The second sum is small simply because $\sum_{n=M}^{\infty} (\frac{4}{5})^n$ can be made small, by choosing $M$ large.

First choose $M$. Then choose $\delta$.

## 2.6 Equicontinuous Families of Functions

## Exercises

1. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set $K$, and $\{f_n\}$ converges pointwise on $K$. Prove that $\{f_n\}$ converges uniformly on $K$.

   Hint: You may assume $K = [a, b]$ is on the real line. Use Theorem 7.25 (Arzela-Ascoli).

2. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann integrable on $[a, b]$, and put

   $$F_n(x) = \int_a^x f_n(t)dt \quad (a \leq x \leq b).$$

   Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

## 2.7 The Stone-Weierstrass Theorem

## Exercises

1. If $f$ is continuous on $[0,1]$ and if
$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, ...),$$
prove that $f(x) = 0$ on $[0,1]$.
Hint: The integral of the product of $f$ with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)dx = 0$.

2. Let
$$\mathcal{C} := \left\{ H : [0,1] \times [0,1] \to \mathbb{R} \mid H(x,y) = \sum_{i=1}^{N} f_i(x)g_i(y) \right\}$$
for some arbitrary $N$ and some arbitrary continuous functions $f_i, g_i : [0,1] \to \mathbb{R}$. Use the Stone-Weierstrass Theorem (on $K = [0,1]^2$) to prove that for each continuous function $F : [0,1]^2 \to \mathbb{R}$ and each $\epsilon > 0$, there exists $H \in \mathcal{C}$ such that
$$\sup_{x,y\in[0,1]} |F(x,y) - H(x,y)| < \epsilon.$$

3. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous at each point. Prove that there is a sequence $P_n$ of polynomials on $\mathbb{R}$ such that for each compact subset $k \subset \mathbb{R}$, the sequence $P_n$ converges uniformly to $f$ on $K$.

   Note: Most continuous functions $f$ on $\mathbb{R}$ cannot be uniformly approximated by polynomials **on the whole** $\mathbb{R}$. Each polynomial, unless constant, will blow up at both $\pm\infty$. Thus, if $f$ is bounded, then $\sup_{x\in\mathbb{R}} |f(x) - p(x)| = \infty$, for each non constant polynomial $p$. In fact, the only $f$ with such uniform approximation must be a polynomial itself. Indeed, if $\sup_{x\in\mathbb{R}} |f(x) - p_n(x)| \to 0$, then $\sup_{x\in\mathbb{R}} |p_{n+1}(x) - p_n(x)| \to 0$, so all polynomials $p_{n+1} - p_n$ are constants.

# Chapter 3

# Some Special Functions

## 3.1 Power Series

### Exercises

1. Prove that

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable infinitely many times and that all derivatives at $x = 0$ are equal to zero, following these steps. This shows that $f$ is not analytic at $x = 0$.

   (a) The function is obviously differentiable at $x \neq 0$. Prove first using induction that for each $x \neq 0$, $f^{(n)}(x) = e^{-\frac{1}{x^2}} P_n(\frac{1}{x})$, for some polynomial $P_n$, whose exact value is irrelevant.

   (b) Prove that

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$$

   for each $m \geq 0$. You may want to use l'Hospital, but after making the change of variables $y = \frac{1}{x}$.

   (c) Use (a), (b) and induction in $n$ to prove that $f^{(n)}(0) = 0$ for each $n \geq 0$.

2. Prove that for each positive integer $p$

$$\sum_{\substack{n+m=p \\ n.m \geq 0}} \frac{1}{n!m!} = \frac{2^p}{p!}.$$

Hint: Use power series, not the binomial formula. More precisely, use that $e^{2x} = e^x e^x$, write down the power series for each side and identify the coefficients of $x^p$ on both sides.

3. Prove that
$$\lim_{\substack{x\to 1\\ x<1}} \sum_{n\geq 1} \frac{x^n}{n} = \infty.$$

## 3.2 Fourier Series

# Exercises

1. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x+2\pi) = f(x)$ for all $x$.

   (a) Compute the Fourier coefficients of $f$.

   (b) Conclude that
   $$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2} \quad (0 < \delta < \pi).$$

   (c) Deduce from Parseval's theorem that
   $$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}.$$

   (d) Put $\delta = \pi/2$ in (c). What do you get?

2. Assume $f : \mathbb{R} \to \mathbb{C}$ is $2\pi$ periodic, differentiable, and has a continuous derivative $f'$.

   (a) Use integration by parts to find a formula relating the Fourier coefficients $\hat{f}(n)$ and $\hat{f}'(n)$ of $f$ and $f'$.

   (b) Prove that the sequence of functions $S_N(f;x)$ converges uniformly to the function $f$ on the real line $\mathbb{R}$.
   Hint: Use the Weierstrass M-test and part (a).

3. Assume $f : [-\pi,\pi] \to \mathbb{C}$ is continuous. Assume its partial Fourier sums $S_N(x)$ converge uniformly (as $N \to \infty$) to a function $g$ on $[-\pi,\pi]$. Prove that $g(x) = f(x)$ for each

$x \in [-\pi, \pi]$.

Hint: Use the triangle inequality

$$\|f - g\|_2 \leq \|S_N f - g\|_2 + \|S - Nf - f\|_2$$

to argue that $\|f - g\|_2 = 0$. Then prove that $g$ is continuous, so $f - g$ is continuous.

4. (a) Prove that the series

$$\sum_{n \geq 2} \frac{e^{inx}}{\log n}$$

converges for each $x \in [-\pi, \pi] \setminus \{0\}$.

Hint: Use summation by parts. Recall also our computations for the Dirichlet kernel.

(b) Prove that there is no continuous function on $[-\pi, \pi]$ whose Fourier series is

$$\sum_{n \geq 2} \frac{e^{inx}}{\log n}.$$

Comment: You can prove part (b) without solving part (a).

# Chapter 4

# Functions of Several Variables

## 4.1 Linear Transformations

An $n$-**dimensional Euclidean vector** is a collection of $n$ real numbers. The book represents vectors as $\mathbf{x} = (x^1, ..., x^n) \in \mathbb{R}^n$. We will choose to write $x = (x_1, ..., x_n)$. The zero vector is the vector whose components are all 0 and is denoted by $\mathbf{0} = (0, ..., 0)$.
A **linear combination** of the vectors $v_1, ..., v_m \in \mathbb{R}^n$ is a summation of the form

$$c_1 v_1 + ... + c_m v_m$$

where $c_1, c_2, ..., c_m \in \mathbb{R}$. The vectors $v_1, ..., v_m \in \mathbb{R}^n$ are **(linearly) independent** if

$$c_1 v_1 + ... + c_m v_m = 0 \implies c_1 = ... = c_m = 0.$$

Otherwise, they are called **dependent**. A **basis** in $\mathbb{R}^n$ is a collection of vectors $n$ vectors $x_1, ..., x_n \in \mathbb{R}^n$ such that for all vectors $v \in \mathbb{R}^n$, there exist unique scalars $c_1, ..., c_n \in \mathbb{R}$ such that

$$v = c_1 x_1 + ... + c_n x_n.$$

The **standard basis** in $\mathbb{R}^n$ is the collection of $\{e_i\}_{i=1}^n$ with $e_j = (0, ..., 1, ..., 0)$ whose only non-zero entry is a unit in the $j^{th}$ coordinate.

**Example.** The vectors $x_1 = (1, 2)$ and $x_2 = (2, 1)$ form a basis for $\mathbb{R}^2$. To check this, let $v \in \mathbb{R}^2$ be any vector. Then

$$v = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \cdot 1 + c_2 \cdot 2 \\ c_1 \cdot 2 + c_2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The unique scalars $c_1, c_2$ are then found by left multiplying both sides of the equation by the inverse of the $2 \times 2$ matrix:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} v.$$

**Definition.** A mapping $A$ (or $L$): $\mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if

1. $A(x_1 + x_2) = A(x_1) + A(x_2)$,
2. $A(cx) = cA(x)$.

Note that these two conditions can be equivalently expressed as

$$A(c_1x_1 + c_2x_2) = c_1A(x_1) + c_2A(x_2)$$

for all scalars $c_1, c_2$ and all vectors $x_1, x_2$.

**Corollary.** $A(\boldsymbol{0}) = 0$.

A linear transformation $A$ is fully determined by its values on a basis.

**Example.** Let $e_1, ..., e_n$ be the standard basis in $\mathbb{R}^n$ and $\tilde{e}_1, ..., \tilde{e}_m$ be the standard basis in $\mathbb{R}^m$. Say I know $A(e_1), A(e_2), ..., A(e_n)$. We would like to be able to find $A(x)$ for any vector $x = x_1e_1 + ... + x_ne_n$ $(x_1, ..., x_n \in \mathbb{R})$. By writing

$$A(e_1) = \sum_{i=1}^{m} a_{i1}\tilde{e}_i,$$

$$...$$

$$A(e_n) = \sum_{i=1}^{m} a_{in}\tilde{e}_i,$$

we find that $a_{ij}$ is known for $1 \leq i \leq m$, $1 \leq j \leq n$. So

$$\begin{aligned} A(x) &= x_1A(e_1) + ... + x_nA(e_n) \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij}\tilde{e}_i \\ &= \sum_{i=1}^{m} \tilde{e}_i \sum_{j=1}^{n} a_{ij}x_j. \end{aligned}$$

**Definition.** The **operator (or algebra)** norm of $A$ is

$$||A|| = \sup\{|Ax| : |x| \leq 1\}.$$

**Theorem 9** (Properties of the algebra norm.). *1.* $|Ax| \leq ||A|| \cdot |x|$ *for all* $x \in \mathbb{R}^n$,

*2.* $|Ax - Ay| = |A(x-y)| \leq ||A|| \cdot |x-y|$,

*3.* $||A|| < \infty$,

*4.* $||A+B|| \leq ||A|| + ||B||$,

*5. If* $B : \mathbb{R}^m \to \mathbb{R}^p$, *then* $||BA|| \leq ||B|| \cdot ||A||$.

*Proof.* 1. This is equivalent to proving

$$\frac{|Ax|}{|x|} \leq ||A||.$$

The LHS of this inequality may be rewritten

$$\frac{|Ax|}{|x|} = \left|\frac{Ax}{|x|}\right| = \left|A\frac{x}{|x|}\right|.$$

But now it follows from the definition of the operator norm that

$$\left|A\frac{x}{|x|}\right| \leq ||A||.$$

2. Equivalently, $d(Ax - Ay) \leq ||A|| \cdot d(x,y)$. Follows from (1).

3. We have that

$$\begin{aligned}
|Ax| &= \sqrt{\left(\sum_{j=1}^{n} a_{1j}x_j\right)^2 + \ldots + \left(\sum_{j=1}^{n} a_{mj}x_j\right)^2} \\
&\leq \sqrt{\sum_{j=1}^{n} a_{1j}^2 \sum_{j=1}^{n} x_j^2 + \ldots + \sum_{j=1}^{n} a_{mj}^2 \sum_{j=1}^{n} x_j^2} \\
&= \sqrt{\sum_{j=1}^{n} x_j^2 \sum_{i,j} a_{ij}^2} \\
&= |x|\sqrt{\sum_{\text{all } i,j} a_{ij}^2}.
\end{aligned}$$

Now, when $|x| \leq 1$, we have that $|Ax| \leq \sqrt{\sum a_{ij}^2}$. So

$$||A|| \leq \sqrt{\sum_{\text{all } i,j} a_{ij}^2} < \infty.$$

4.

5.

$\square$

## 4.2 Differentiation

A real valued function of a real variable is differentiable at $x_0 \in \mathbb{R}$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and we denote the value of the limit by $f'(x_0)$. The existence of this limit is equivalent to

$$\begin{aligned} &\iff \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \\ &\iff \lim_{x \to x_0} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} = 0. \end{aligned}$$

We can think of the product of the derivative $f'(x_0)$ and the quantity $x - x_0$ as the application of a linear operator $A : \mathbb{R} \to \mathbb{R}$ to $x - x_0$. If $h \in \mathbb{R}$ is an arbitrary real number, then $Ah = hf'(x_0)$. Through the change of variables $h = x - x_0$, we may restate the definition of differentiation as follows: a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = x_0$ if there exists a linear map $A : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - Ah|}{|h|} = 0.$$

**Definition.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and suppose $x_0 \in \mathbb{R}^n$. We say that $f$ is **differentiable** at $x_0$ if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - Ah|}{|h|} = 0.$$

If such a mapping $A$ exists, we call it $f'(x_0)$, the differential of $f$ at $x_0$.

**Corollary.** *If it exists, $A$ is unique.*

*Proof.* Let $A_1, A_2$ be two such linear maps. Then

$$|A_1 h - A_2 h| \leq |f(x_0 + h) - f(x_0) - A_1 h| + |f(x_0 + h) - f(x_0) - A_2 h|$$

by the triangle inequality. Dividing by $|h|$ and taking limits, we have

$$\lim_{h \to 0} \frac{|A_1 h - A_2 h|}{|h|} \leq \lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - A_1 h|}{|h|} + \lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - A_2 h|}{|h|}.$$

Because both terms on the RHS approach 0 as $h \to 0$, we can conclude that

$$\lim_{h \to 0} \frac{|A_1 h - A_2 h|}{|h|} = 0.$$

We want to prove that $A_1 x = A_2 x$, or equivalently, $|A_1 x - A_2 x| = 0$ for all $x$. Fix $x$ and let $h \to 0$ by considering those vectors $h$ which are increasingly smaller multiples of $x$. Then we may write

$$x = h\frac{|x|}{|h|}.$$

Then

$$|A_1 x - A_2 x| = |A_1 h - A_2 h|\frac{|x|}{|h|} = \frac{|A_1 h - A_2 h|}{|h|}|x| \longrightarrow 0$$

as $h \to 0$. So $A_1 = A_2$. □

**Theorem 10** (Chain Rule). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^p$ and consider $x_0 \in \mathbb{R}^n$. Call $y_0 = f(x_0)$. Assume $f$ is differentiable at $x_0$ and $g$ is differentiable at $y_0$. Then $F = g \circ f$ is differentiable at $x_0$ and*

$$F'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

*Proof.* We need to prove

$$\frac{F(x_0 + h) - F(x_0) - BA(h)}{|h|} \longrightarrow 0.$$

Call $u(h) = f(x_0 + h) - f(x_0) - Ah$ and $\varepsilon(h) = \frac{u(h)}{|h|}$. Then our goal amounts to showing that $\varepsilon(h) \to 0$. □

# Exercises

1. Assume $F : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at the origin. Prove that

$$\lim_{n\to\infty} n\left[F\left(\frac{1}{n}, \frac{3}{n}\right) + F\left(\frac{3}{n}, \frac{1}{n}\right) - 2F\left(\frac{2}{n}, \frac{2}{n}\right)\right]$$

exists and find it.

2. Prove that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

   (a) is continuous at $(0,0)$ (use $\epsilon, \delta$; for each $\epsilon > 0$, find $\delta > 0$ such that $|(x,y)-(0,0)| = \sqrt{x^2+y^2} < \delta$ implies $|f(x,y)| < \epsilon$)

(b) has partial derivatives at $(0,0)$

(c) is not differentiable at $(0,0)$.

3. Suppose that $F$ and $G$ are maps from $\mathbb{R}^n$ into $\mathbb{R}$, differentiable at some point $x_0 \in \mathbb{R}^n$, and that $F(x_0) = G(x_0)$. Next let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose that $F(x) \le f(x) \le G(x)$ for all $x$ in some neighborhood $V$ of $x_0$. Prove that $f$ is differentiable at $x_0$.

4. Suppose that $f$ is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f$,...,$D_n f$ are bounded in $E$. Prove that $f$ is continuous in $E$.
Hint: Proceed as in the proof of Theorem 9.21.

5. See Def. 9.39. Put $f(0,0) = 0$ and
$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$
if $(x,y) \neq (0,0)$. Prove that

(a) $f$, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$;

(b) $D_{12} f$ and $D_{21} f$ exist at every point of $\mathbb{R}^2$, and are continuous except at $(0,0)$;

(c) $(D_{12} f)(0,0) = 1$ and $(D_{21} f)(0,0) = -1$.

6. Assume $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable everywhere and that
$$|D_2 f_1(x,y)| \le \frac{1}{8}$$
$$|D_1 f_2(x,y)| \le \frac{1}{8}$$
$$|D_1 f_1(x,y)| \le \frac{1}{8}$$
$$|D_2 f_2(x,y)| \le \frac{1}{8}$$
for each $(x,y) \in \mathbb{R}^2$. Show that $f$ is one to one on $\mathbb{R}^2$.
Hint: Do a similar argument to the one in Theorem 9.24 part (a).

## 4.3 The Contraction Principle

## Exercises

1. For this problem, it may be useful to use the inequality
$$\frac{x+y}{2} \ge \sqrt{xy}$$

valid for all positive real numbers $x, y$.

Let now $A, x_1$ be numbers $> 0$. Define the sequence $x_n$ via the recurrence:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right), \quad n \geq 1.$$

Use the Contraction Principle to prove the convergence of $x_n$. Recall that this principle (or rather its proof) says that the sequence defined by $x_{n+1} = Tx_n$ converges, whenever $T$ is a contraction.

Hint: Show that the map $T(x) = \frac{1}{2}(x + \frac{A}{x})$ maps $[\sqrt{A}, \infty)$ into $[\sqrt{A}, \infty)$. Show that $T$ is a contraction in this space, using the Mean Value Theorem. You may take for granted that $[\sqrt{A}, \infty)$ is a complete metric space (and here is a quick proof of that: take a Cauchy sequence $x_n$ in $[\sqrt{A}, \infty)$. This sequence is obviously a sequence of real numbers, and since $\mathbb{R}$ is complete, the sequence must have a limit. Since $x_n \geq \sqrt{A}$, the limit should also be in the space $[\sqrt{A}, \infty]$. So the sequence converges to a limit in $[\sqrt{A}, \infty)$, making this space complete).

## 4.4 The Inverse Function Theorem

## Exercises

1. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$$

   (a) What is the range of $\mathbf{f}$?

   (b) Show that the Jacobian of $\mathbf{f}$ is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which $f$ is one-to-one. Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$.

   (c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$ compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52) in Rudin §9.

## 4.5 The Implicit Function Theorem

## Exercises

1. Define the set

$$X = \{(x_1, x_2, x_2, x_4) \in \mathbb{R} : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2, \; x_1 + x_2 + x_3 + x_4 = 2\}.$$

   (a) Prove that each point $x$ in $X$ other than $(1, 1, 1, -1)$ has a neighborhood $U$, such that $X \cap U$ is the graph of a function in two of the variables $x_1, ..., x_4$ (which variables exactly will depend on the point). Equivalently, this means that there is a choice of two indices $\{i_1, i_2\} \subset \{1, 2, 3, 4\}$ (and this choice depends only on $x$) such that each $y = (y_1, y_2, y_3, y_4) \in U \cap X$ is uniquely determined by its coordinates $y_{i_1}, y_{i_2}$.

   (b) Prove that (a) is false for $x = (1, 1, 1, -1)$.